10.2 Sequences

this section is not on exam 2

sequence: list of numbers in some order

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}$$

sequence **converges** if $\lim_{n\to\infty} a_n$ exists

for example, $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots, \frac{99{,}999}{100{,}000}, \ldots\right\}$

the terms appear to approach 1 $\qquad \lim_{n\to\infty} \frac{n}{n+1} = 1$

so $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ converges to 1 example

$\left\{ n^{13/n} \right\}_{n=1}^{\infty}$ converges?

$= \left\{ 1^{13}, 2^{13/2}, 3^{13/3}, \ldots, 13^{13/13}, \ldots \right\}$

$\quad 1 \qquad\qquad\qquad\qquad 13$

the terms appear to initially get bigger

does the sequence converge?

listing numbers does not seem to answer that question

try $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} n^{13/n} \to \infty^0$ indeterminate

limit could be anything

l'Hospital's Rule can give us the limit

let $y = n^{13/n}$ $\longrightarrow$ we want $\lim\limits_{n\to\infty} y$

$\ln y = \ln n^{13/n} = \frac{13}{n}\ln(n) = \frac{13\ln(n)}{n} \to \frac{\infty}{\infty}$ as $n\to\infty$

we will use l'Hospital's Rule on this

$$\lim_{n\to\infty} \frac{13\ln(n)}{n}$$

$$= \lim_{n\to\infty} \frac{13\cdot\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{13}{n} = 0$$

this means $\lim_{n\to\infty} \ln y = 0$ but we want $\lim_{n\to\infty} y$ where $y = n^{13/n}$

remember $e^{\ln y} = y$

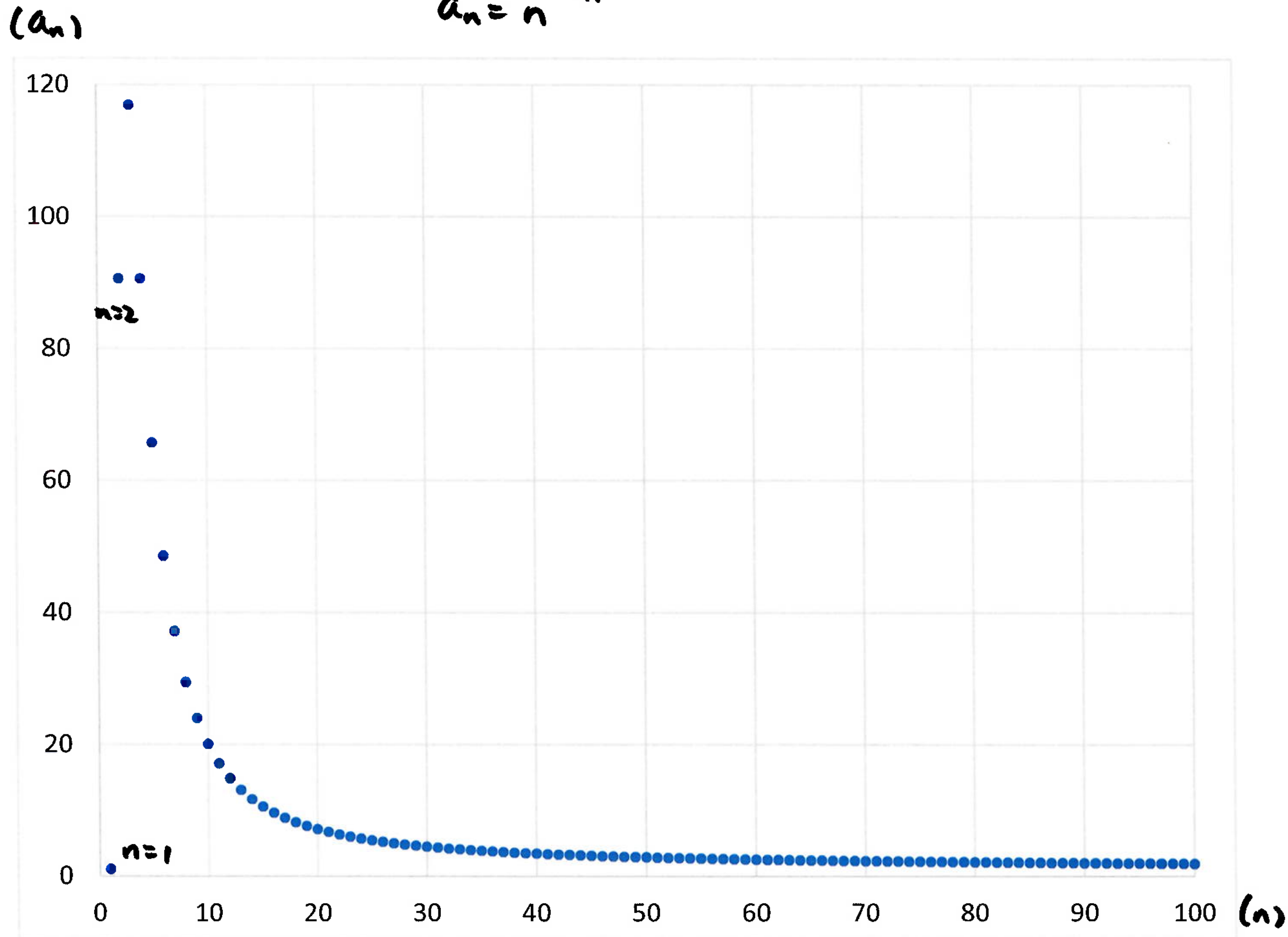
so, $\lim_{n\to\infty} \ln y = 0$ becomes

$\lim_{n\to\infty} e^{\ln y} = e^0 = \boxed{1}$ this is the limit of $\{n^{13/n}\}_{n=1}^{\infty}$

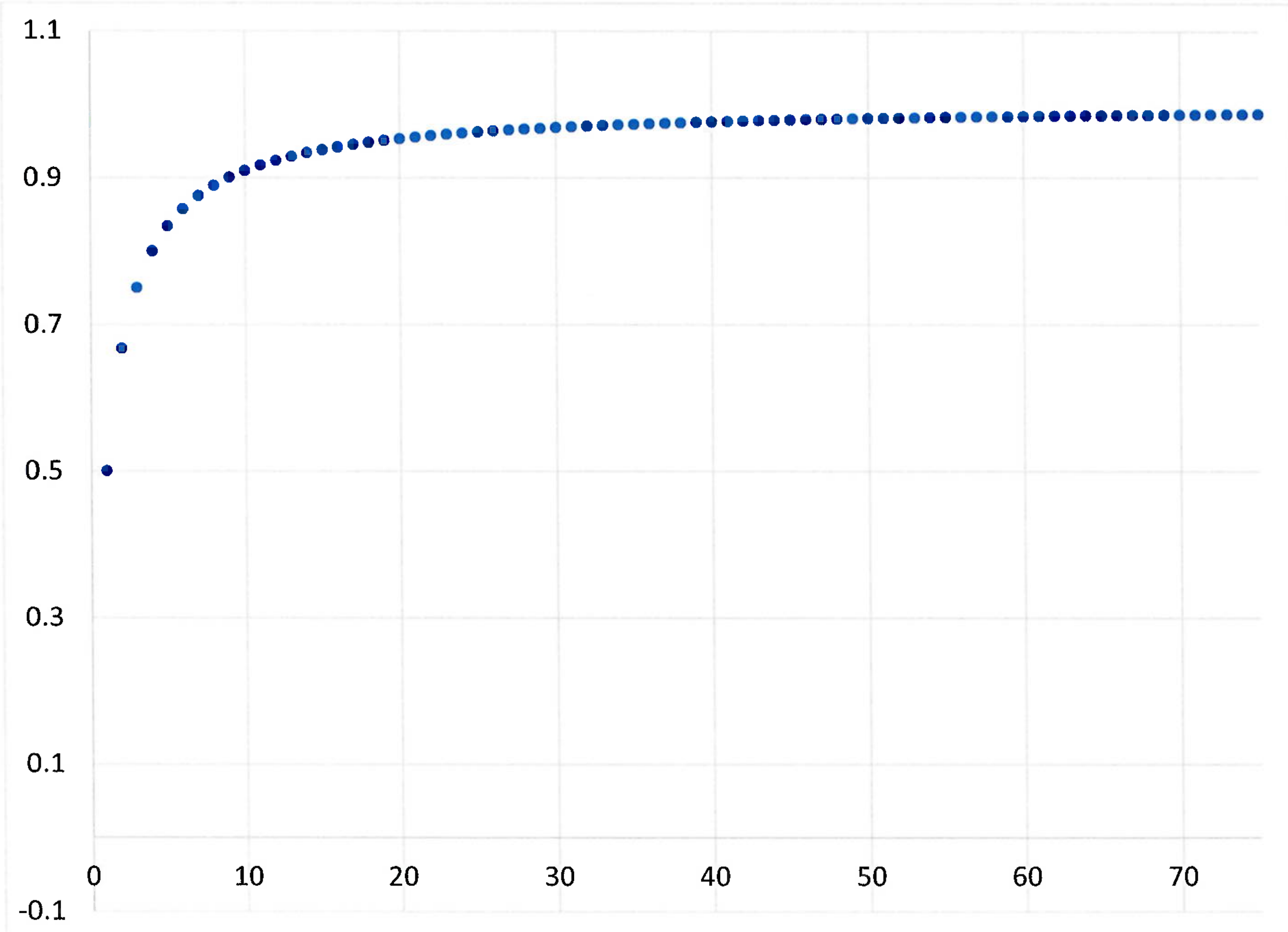
$$\{1, 2^{13/2}, 3^{13/3}, \dots, 1{,}000{,}000^{13/1000000}, \dots\}$$

wait long enough, very close to 1

$a_n = n^{13/n}$



$\left\{ \frac{n}{n+1} \right\}$



$\{n^{13/n}\}$ increases initially, then decreases and settles down around 1

whereas $\left\{\frac{n}{n+1}\right\}$ always increases

if a sequence either always increases or always decreases (for all n)

then the sequence is said to be monotonic

$\{1, 2, 3, 4, 5, \ldots\}$ is monotonic

$\{-1, -2, -3, -5, -7, -9, -13, \ldots\}$ is monotonic

$\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots\}$ is not monotonic

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} = \left\{\frac{1}{2}, \underbrace{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots}_{\text{increases to approach } 1}\right\}$$

Smallest term in this sequence

this sequence is bounded because ~~no~~ all terms are no smaller than something $\left(\frac{1}{2}\right)$ and no bigger than something else (1)

if a sequence is bounded and monotonic, then it ~~is~~ converges

An important sequence is the <u>geometric sequence</u> $\rightarrow \{r^n\}$ r is constant

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots\right\}$$

$$= \left\{\left(\tfrac{1}{2}\right)^1, \left(\tfrac{1}{2}\right)^2, \left(\tfrac{1}{2}\right)^3, \left(\tfrac{1}{2}\right)^4, \dots\right\} = \left\{\left(\tfrac{1}{2}\right)^n\right\}_{n=1}^{\infty}$$ converges to 0

↳ $r = \frac{1}{2}$

common ratio between succeeding terms

$$\left\{-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \dots\right\} = \left\{\left(-\tfrac{1}{3}\right)^n\right\}_{n=1}^{\infty}$$ converges to 0

$$\{2, 4, 8, 16, 32, 64, \dots\} = \{2^n\}_{n=1}^{\infty}$$ diverges

the ratio r appears to determine convergence

the geometric sequence $\{r^n\}$ converges if $-1 < r \leq 1$

why is 1 included by -1 is not?

$$\{r^n\}_{n=1}^{\infty}$$

if $r=1$: $\{1, 1, 1, 1, 1, 1, \dots\}$ converges to 1

if ~~r=1~~ $r=-1$: $\{-1, 1, -1, 1, -1, 1, \dots\}$ no limit, diverges factorial (!) shows up a lot in sequences and series

$n! = (n)(n-1)(n-2)\cdots(1)$

$4! = 4\cdot 3\cdot 2\cdot 1 = 24$

$5! = 5\cdot 4\cdot 3\cdot 2\cdot 1 = 120$

<u>example</u> $\left\{\dfrac{n}{n!}\right\}_{n=1}^{\infty}$

$= \left\{1, \dfrac{2}{2!}, \dfrac{3}{3!}, \dfrac{4}{4!}, \ldots\right\}$

$= \left\{1, 1, \dfrac{1}{2}, \dfrac{1}{6}, \ldots\right\}$

converges? check $\displaystyle\lim_{n\to\infty} \frac{n}{n!}$

$n\to\infty$ numerator is large

denominator is very LARGE

limit appears to be zero

let's check properly : $\displaystyle\lim_{n\to\infty} \frac{n}{n!} = \lim_{n\to\infty} \frac{\cancel{n}}{(\cancel{n})(n-1)(n-2)\cdots(1)}$

$$= \lim_{n\to\infty} \frac{1}{(n-1)(n-2)\cdots(1)} = 0 \qquad \text{so, } \left\{\frac{n}{n!}\right\} \text{ converges to } 0$$

<u>example</u> $\left\{ \dfrac{e^n}{n!} \right\}_{n=1}^{\infty}$

$$= \left\{ \frac{e^1}{1!}, \frac{e^2}{2!}, \frac{e^3}{3!}, \dots \right\}$$

initially, e^n dominates $n!$

as n becomes large, for example

$n=10$: $e^{10} \approx 22{,}026$

$10! \approx 3{,}628{,}800$

$n=100$: $e^{100} \approx 3 \times 10^{43}$

$100! \approx 9 \times 10^{157}$

$n\to\infty$, denominator wins, limit appears to be zero can we be more sure?

$$\frac{e^n}{n!} = \frac{e^n}{(n)(n-1)(n-2)\cdots(1)}$$

as n becomes large, the denominator is polynomial w/ leading term n^n
which dominates

so, as $n \to \infty$, $\frac{e^n}{n!} \approx \frac{e^n}{n^n} = \left(\frac{e}{n}\right)^n$ looks like r^n
with $r \to 0$ as $n \to \infty$
and since $-1 < r \leq 1$
this behaves like a
geo. seq. as $n \to \infty$
therefore converges.